

G.3 Notes \Rightarrow Gram-Schmidt & QR-Decomposition

★ The key idea to keep in mind throughout this section is projections. (You may want to review your notes from Calc D to help.)

Orthogonal Set \Rightarrow Vectors are all orthogonal to each other.
Orthonormal Set \Rightarrow A set of orthogonal unit vectors.

Thm. 6.3.1: $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthonormal basis for V and \vec{u} is any vector in V , then
$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \dots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

The "coordinates" of \vec{u} relative to S (the coefficients of the \vec{v}_i 's) are just the inner products — but these inner products are just the projections!

Recall: $\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$

But now we're using a more general inner product: $\langle \vec{u}, \vec{v} \rangle$ and since S is an orthonormal set, $\|\vec{v}\| = 1$.

So, Thm. 6.3.1 just says that a vector \vec{u} in V can be broken up into components following each basis vector. The length of each component is the inner product. (Look at equation (1) on page 301. I would have put this before Thm. 6.3.1.)

Thm. 6.3.5 explains how to find the projection of a vector onto an entire vector space W . When we did projections before, we only projected vectors onto vectors. Now we can project vectors onto an entire space — all you need to do is project the vector onto each of the orthogonal (or orthonormal) basis vectors and add.

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The Gram-Schmidt process lets us convert a basis for a space into an orthonormal basis.

Here's the basic idea:

- Start with the first vector. ($\vec{v}_1 = \vec{u}_1$)
- To create a second vector (\vec{v}_2) orthogonal to the first one, let $\vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1} \vec{u}_2$.
See, we're only interested in the part of \vec{u}_2 that's orthogonal to \vec{u}_1 (the other stuff is already accounted for in $\vec{v}_1 = \vec{u}_1$).
- So, now we have two orthogonal basis vectors. To create the third orthogonal vector, we only care about the "stuff" in \vec{u}_3 that is not accounted for with \vec{v}_1 and \vec{v}_2 . In other words, we want the component of \vec{u}_3 that is orthogonal to the space spanned by \vec{v}_1 and \vec{v}_2 . That means we want \vec{u}_3 minus the part that would be in the span of \vec{v}_1 and \vec{v}_2 . W_2 is the space spanned by \vec{v}_1 and \vec{v}_2 . So,

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \text{proj}_{W_2} \vec{u}_3 \\ &= \vec{u}_3 - (\text{proj}_{\vec{v}_1} \vec{u}_3 + \text{proj}_{\vec{v}_2} \vec{u}_3) \\ &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2\end{aligned}$$

etc. So, $\vec{v}_4 = \vec{u}_4 - \text{proj}_{W_3} \vec{u}_4$

$$= \vec{u}_4 - \text{proj}_{\vec{v}_1} \vec{u}_4 - \text{proj}_{\vec{v}_2} \vec{u}_4 - \text{proj}_{\vec{v}_3} \vec{u}_4$$

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After n steps of this, you have $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ that are all orthogonal to each other.

To create an orthonormal set, just normalize each vector \vec{v}_i (i.e., divide each by its own magnitude.)

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}, \quad \text{etc.}$$

QR-Decomposition

QR-Decomposition is a type of matrix division. The goal is to take a matrix A and factor it into two other matrices Q and R , where Q has orthonormal columns and R is upper triangular.

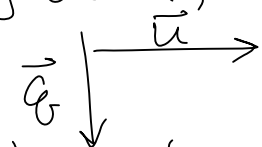
The steps for QR-Decomposition are as follows:

- Take the column vectors of A and do the Gram-Schmidt process on them to create an orthonormal set. These new vectors are the columns of Q .

- To get R , do a bunch of inner products between the \vec{u} 's (the columns of A) and the \vec{q} 's.

To help remember: R is upper triangular, the \vec{q} 's increase going down, and the \vec{u} 's increase going across

(“qu” like in spelling and $m \times n$ like rows then columns.)



Kinda silly, but it helps me remember.

-Abby Brown 5/5/04