6.3 Notes ⇒ Gram-Schmidt & QR-Decomposition

☆ The key idea to keep in mind throughout this section is <u>projections</u>. (You may want to review your notes from Calc D to help.)

Orthogonal Set ⇒ Vectors are all orthogonal to each other.
Orthonormal Set ⇒ A set of orthogonal <u>unit</u> vectors.

Thm. 6.3.1: $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis for V and $\vec{u}$ is any vector in V, then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

The "coordinates" of $\vec{u}$ relative to S (the coefficients of the $\vec{v}_i$'s) are just the inner products — but these inner products are just the projections!

Recall: $\text{proj}_{\vec{v}}\, \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$

But now we're using a more general inner product: $\langle \vec{u}, \vec{v} \rangle$ and since S is an ortho<u>normal</u> set, $\|\vec{v}\| = 1$.

So, Thm. 6.3.1 just says that a vector $\vec{u}$ in V can be broken up into components following each basis vector. The length of each component is the inner product. (Look at equation (1) on page 301. I would have put this before Thm. 6.3.1.)

Thm. 6.3.5 explains how to find the projection of a vector onto an entire vector space W. When we did projections before, we only projected vectors onto vectors. Now we can project vectors onto an entire space — all you need to do is project the vector onto each of the orthogonal (or orthonormal) basis vectors and add.

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The **Gram-Schmidt process** lets us convert a basis for a space into an orthonormal basis.

Here's the basic idea:

- Start with the first vector. ($\vec{v}_1 = \vec{u}_1$)
- To create a second vector ($\vec{v}_2$) orthogonal to the first one, let $\vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1}\vec{u}_2$.
  See, we're only interested in the part of $\vec{u}_2$ that's orthogonal to $\vec{u}_1$ (the other stuff is already accounted for in $\vec{v}_1 = \vec{u}_1$).
- So, now we have two orthogonal basis vectors.
  To create the third orthogonal vector, we only care about the "stuff" in $\vec{u}_3$ that is not accounted for with $\vec{v}_1$ and $\vec{v}_2$. In other words, we want the component of $\vec{u}_3$ that is orthogonal to the space spanned by $\vec{v}_1$ and $\vec{v}_2$. That means we want $\vec{u}_3$ minus the part that would be in the span of $\vec{v}_1$ and $\vec{v}_2$. W_2 is the space spanned by $\vec{v}_1$ and $\vec{v}_2$. So,

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \text{proj}_{W_2}\vec{u}_3 \\ &= \vec{u}_3 - (\text{proj}_{\vec{v}_1}\vec{u}_3 + \text{proj}_{\vec{v}_2}\vec{u}_3) \\ &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2\end{aligned}$$

etc. So, $\vec{v}_4 = \vec{u}_4 - \text{proj}_{W_3}\vec{u}_4$

$$= \vec{u}_4 - \text{proj}_{\vec{v}_1}\vec{u}_4 - \text{proj}_{\vec{v}_2}\vec{u}_4 - \text{proj}_{\vec{v}_3}\vec{u}_4$$

(continued)

After n steps of this, you have $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ that are all orthogonal to each other.

To create an orthonormal set, just normalize each vector $\vec{v}_i$ (i.e., divide each by its own magnitude.)

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}, \text{ etc.}$$

QR-Decomposition

QR-Decomposition is a type of matrix division. The goal is to take a matrix A and factor it into two other matrices Q and R, where Q has orthonormal columns and R is upper triangular.

The steps for QR-Decomposition are as follows:

- Take the column vectors of A and do the Gram-Schmidt process on them to create an orthonormal set. These new vectors are the columns of Q.

- To get R, do a bunch of inner products between the $\vec{u}$'s (the columns of A) and the $\vec{q}$'s.

  To help remember: R is upper triangular, the $\vec{q}$'s increase going down, and the $\vec{u}$'s increase going across ("qu" like in spelling and mxn like rows then columns.) $\vec{u}$ $\vec{q}$

  Kinda silly, but it helps me remember.

–Abby Brown 5/5/04