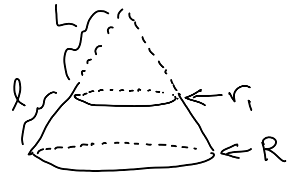
AREA OF A SURFACE OF REVOLUTION (page 1 of 2)

First we need to find the lateral surface area of the frustum of a cone.



Lateral surface area of large cone

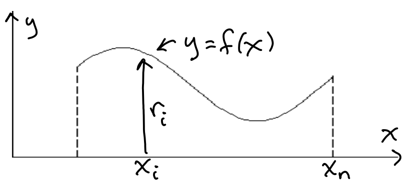
Lat. surf. area of top cone

$$\text{Surface Area of Frustum} = \pi R(\ell + L) - \pi r_1 L = \pi((R - r_1)L + R\ell)$$

$$\text{similar triangles} \Rightarrow \frac{L}{r_1} = \frac{\ell + L}{R} \Rightarrow LR = r_1\ell + r_1 L \Rightarrow L(R - r_1) = r_1\ell$$

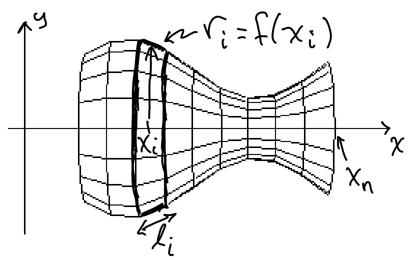
$$\therefore \text{Surf. Area of Frustum} = \pi(r_1\ell + R\ell). \quad \text{Let } r = \text{avg. radius} = \frac{r_1 + R}{2}$$

$$\boxed{\text{Surf. Area of Frustum} = 2\pi r\ell}$$



Given a surface of revolution, imagine breaking it into strips. Each strip can be approximated as a narrow frustum.

about the x-axis



The lateral surface area of strip $i = 2\pi r_i \ell_i$

$\ell_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i)$ using approx. for arc length.

$$\therefore \text{Area approximation} = \sum_{i=1}^{n} 2\pi r_i \ell_i = \sum_{i=1}^{n} 2\pi \overbrace{f(x_i)}^{r}\,\overbrace{\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i}^{\ell}$$

Again, we need to take the limit as $\Delta x_i \to 0$ or $n \to \infty$ for <u>exact</u> answer.

upper-case

$$\text{Area of Surface of Revolution about the x-axis} = S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

for $y = f(x)$ on $[a, b]$.

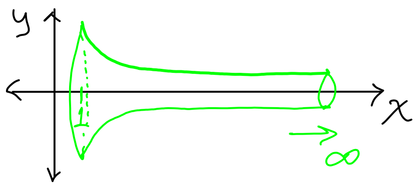
Note that this is a general sketch and not a formal proof.

Keeping $A = 2\pi r\ell$ in mind, we can develop four cases.

Area of a Surface of Revolution – Page 2

	curve: $y=f(x)$ on $[a,b]$ use dx	curve: $x=f(y)$ on $[c,d]$ use dy
Rotation about x-axis	$r=f(x)$ $S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$	$r=y$ $S=2\pi\int_c^d y\sqrt{1+(f'(y))^2}\,dy$
Rotation about y-axis	$r=x$ $S=2\pi\int_a^b x\sqrt{1+(f'(x))^2}\,dx$	$r=f(y)$ $S=2\pi\int_c^d f(y)\sqrt{1+(f'(y))^2}\,dy$

Ex: "Gabriel's Horn" Region $y=\frac{1}{x}$ ($f(x)$), $y=0$, $x\geq 1$
Revolved about the x-axis.



$$S=2\pi\int_1^\infty \underbrace{\frac{1}{x}}_{f(x)}\sqrt{1+\underbrace{\left(\frac{-1}{x^2}\right)^2}_{(f'(x))^2}}\,dx$$

$$=2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$$

Since $\sqrt{x^4+1}>\sqrt{x^4}$,

$$S>2\pi\int_1^\infty \frac{x^2}{x^3}\,dx=2\pi\int_1^\infty \frac{1}{x}\,dx=2\pi\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

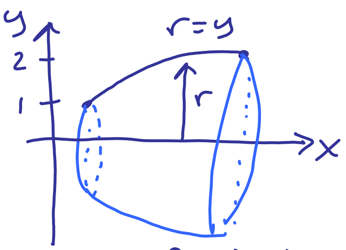
$$=2\pi\lim_{b\to\infty}\ln|x|\Big|_1^b$$

$$=2\pi\left(\lim_{b\to\infty}\ln b-\ln 1\right)$$

$$=2\pi(\infty-0)=\infty \quad \text{Diverges}$$

$\therefore$ S also diverges since $S>$ ⟶ $\Rightarrow$ Infinite surface area

Ex: $x=1+2y^2$, $1\leq y\leq 2$ rotated about the x-axis.



$x=f(y)=1+2y^2$
$f'(y)=4y$

$$S=2\pi\int_c^d y\sqrt{1+(f'(y))^2}\,dy$$

$$=2\pi\int_1^2 y\sqrt{1+(4y)^2}\,dy$$

$$=2\pi\int_1^2 y\sqrt{1+16y^2}\,dy$$

$u=16y^2+1$
$du=32y\,dy$

$$=\frac{2\pi}{32}\int_{17}^{65}u^{1/2}\,du$$

$$=\frac{2\pi}{32}\cdot\frac{2}{3}u^{3/2}\Big|_{17}^{65}=\boxed{\frac{\pi}{24}\left(65\sqrt{65}-17\sqrt{17}\right)}$$