Taylor Polynomials and Series (page 1 of 5)

Consider $f(x) = e^x$.
Find a linear approximation for $f(x)$ at $x = 0$.

$f'(x_0)$ is the slope or "1st degree shape of the curve" at $x = x_0$.
Find a quadratic approximation for $f(x)$ at $x = 0$. What needs to match now?

What would you need to do for a cubic approximation?
But let's not show all of that work... at least not yet.
$P_3(x) =$

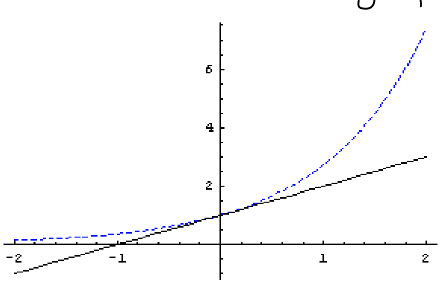
Let's check this numerically: $f(x) = e^x$

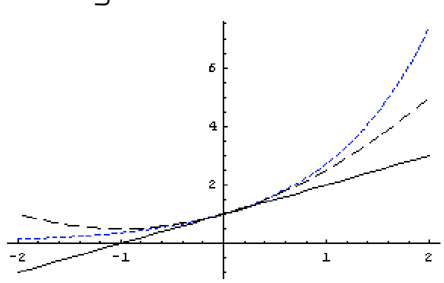
$f(0) =$ $\qquad$ $f(0.25) =$ $\qquad$ $f(0.5) =$ $\qquad$ $f(1) =$

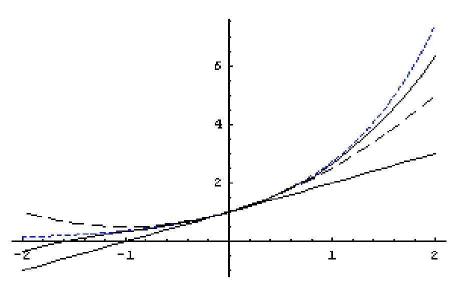
$P_3(0) =$ $\qquad$ $P_3(0.25) =$ $\qquad$ $P_3(0.5) =$ $\qquad$ $P_3(1) =$

What happens as we get further from zero?

Let's check this graphically: $f(x) = e^x$







$f(x) =$ $\qquad$, $P_1(x) =$ $\qquad$ $P_2(x) =$ $\qquad$ $P_3(x) =$

$P_3(x)$ is the "polynomial expansion of 3rd degree about 0."
In general, the n^{th} degree polynomial expansion about $x=a$ is written

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

To find the c_i coefficients, match derivatives.

$$P_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$
$$P_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + \cdots + (n-1)nc_n(x-a)^{n-2}$$
$$P_n'''(x) = 2\cdot 3c_3 + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$$
$$\vdots \leftarrow n^{th} \text{ derivative}$$
$$P_n^{(n)}(x) = (1)(2)\cdot\cdots\cdot(n-3)(n-2)(n-1)nc_n = n!\,c_n$$

Match derivatives at $x=a$ ← the "center" = point about which we expand the function.

$f(a) = P_n(a) =$ $\qquad$ $f''(a) = P_n''(a) =$

$f'(a) = P_n'(a) =$ $\qquad$ $f^{(n)}(a) = P_n^{(n)}(a) =$

$\Rightarrow c_n =$

If f has n derivatives at a, then

☆ The <u>n^{th} Taylor Polynomial for f at a</u> is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

n^{th} derivative

The <u>n^{th} Maclaurin Polynomial for f</u> is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Maclaurin Polynomial = Taylor Polynomial centered at 0 instead of at a.

Ex: The n^{th} Maclaurin Polynomial for $f(x)=e^x$ is

$$T_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

Ex: Find the fourth Taylor Polynomial for $f(x) = \sin x$ expanded about $\pi/4$. Use $T_4(x)$ to approximate $\sin(0.8)$. Compare this to the calculator value for $\sin(0.8)$. What about $\sin(\pi)$?