If f is represented by a power series $f(x)=\sum c_n(x-a)^n$ for all x in an interval I containing a, then $c_n=f^{(n)}(a)/n!$ and

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

This means every convergent power series has the same form.

★ Taylor Series If a function f has derivatives of all orders at $x=a$, then the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ is the Taylor Series for f at a. If $a=0$, then the series is the Maclaurin series for f.
Basically, this means we can let $n\to\infty$ in the polynomial approximations.

Ex: $f(x)=\cos x$, $a=0$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots\\ &= 1+0x+-\tfrac{1}{2}x^2+0x^3+\tfrac{1}{4!}x^4+0x^5+-\tfrac{1}{6!}x^6+\cdots\\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=\sum_{n=0}^{\infty}\end{aligned}$$

Check for interval of convergence.
Ratio Test: $\lim_{n\to\infty}\Big|$

This series converges for the above values of x. But this doesn't answer the new question: Does the Taylor Series converge to $f(x)$? i.e., does $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$?

Define $R_n(x)$ such that $f(x)=T_n(x)+R_n(x)$.

★ $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ if and only if $\lim_{n\to\infty}R_n(x)=0$ for all x in the interval of convergence.

Use the squeeze theorem to show $\lim_{n\to\infty}|R_n(x)|=0$.
Taylor's Inequality ⇒ If $|f^{(n+1)}(x)|\le M$ for all x in the interval of convergence, then

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}$$

$\therefore$ We need to find bounds for $|f^{(n+1)}(x)|$.

In our example, $f(x)=\cos x$, all derivatives are $\pm\sin x$ or $\pm\cos x$.

$\therefore \quad 0 \le |f^{(n+1)}(x)| \le 1 \Rightarrow$ Let $M=1$

Now we have $0 \le |R_n(x)| \le \left|\frac{f^{(n+1)}(x)}{n!} x^{n+1}\right| \le \frac{1}{(n+1)!} x^{n+1}$

Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{(n+1)!} x^{n+1} = 0$, $\lim_{n\to\infty} R_n(x) = 0$ as well.

Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x.

This may not seem very interesting since it directly follows from the approximations we were working with before. However, it is important since it proves that if we let *n* approach infinity, then the series is EQUAL to *f*(*x*) (in the interval of convergence) and not just an approximation!

Composite Functions

If you already know a power series for $f(x)$, then you can find the power series for $f(g(x))$ by substitution.

Example: $f(x) = x\cos 2x \qquad$ Note: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $R=\infty$

$\therefore \cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\quad)^{2n}}{(2n)!}$

$=$

$\therefore x\cos(2x) = x\sum(\qquad\qquad) =$

We can also use Taylor Series to find the sums of some series.

Ex: Find the sum $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} =$ match to $\cos x$ series above.

Ex: Find the Maclaurin series for $f(x) = \ln(1+x)$ and its radius of convergence.