TPHS Linear Algebra — SDSU Math 254   Name: Key

Quiz #5 (6.1 - 6.5)   _____   Per.: _____

Clearly show *all* of your work.   15 pts.

Scientific calculators allowed.

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1) Use the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$$

to compute the following for
$\mathbf{f} = f(x) = x^2$
$\mathbf{g} = g(x) = x^2 + x$

a) $d(\mathbf{f}, \mathbf{g})$ (exactly) (3 pts)
b) the angle between **f** and **g**. (4 pts.) (accurate to three decimal places)

a) $d(\vec{f},\vec{g}) = \|\vec{f}-\vec{g}\| = \langle \vec{f}-\vec{g}, \vec{f}-\vec{g}\rangle^{1/2}$

$\vec{f}-\vec{g} = x^2 - (x^2+x) = -x$

$\langle -x, -x\rangle^{1/2} = \left(\int_0^1 (-x)(-x)\,dx\right)^{1/2}$

$= \left(\int_0^1 x^2\,dx\right)^{1/2} = \left(\frac{1}{3}\right)^{1/2}$

$= \boxed{\frac{\sqrt{3}}{3}}$

b) $\cos\theta = \dfrac{\langle \vec{f}, \vec{g}\rangle}{\|\vec{f}\|\,\|\vec{g}\|}$

$\langle \vec{f}, \vec{g}\rangle = \int_0^1 x^2(x^2+x)\,dx = \int_0^1 x^4 + x^3\,dx$

$= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$

$\|\vec{f}\| = \langle \vec{f}, \vec{f}\rangle^{1/2} = \left(\int_0^1 (x^2)(x^2)\,dx\right)^{1/2}$

$= \left(\int_0^1 x^4\right)^{1/2} = \left(\frac{1}{5}\right)^{1/2} = \frac{1}{\sqrt{5}}$

$\|\vec{g}\| = \langle \vec{g}, \vec{g}\rangle^{1/2} = \left(\int_0^1 (x^2+x)(x^2+x)\,dx\right)^{1/2}$

$= \left(\int_0^1 x^4 + 2x^3 + x^2\,dx\right)^{1/2}$

$= \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right)^{1/2} = \left(\frac{31}{30}\right)^{1/2}$

$\cos\theta = \dfrac{9/20}{\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{31}}{\sqrt{30}}} = \frac{9}{20}\cdot\sqrt{5}\cdot\frac{\sqrt{30}}{\sqrt{31}} = \frac{9\sqrt{6}}{4\sqrt{31}}$

$$\boxed{\begin{aligned}\theta = \arccos\left(\frac{9\sqrt{6}}{4\sqrt{31}}\right) &\approx 8.163^\circ \\ &\approx 0.142 \text{ radians}\end{aligned}}$$

2) Find the *QR*-Decomposition of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{(7 pts)}$$

$\vec{u}_1 = (1,1,0)$, $\vec{u}_2 = (1,2,0)$, $\vec{u}_3 = (0,1,2)$

$\vec{v}_1 = \vec{u}_1 = (1,1,0)$ $\|\vec{v}_1\| = \sqrt{2}$

$\vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1}\vec{u}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

$= (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2}, \frac{1}{2}, 0)$

$\|\vec{v}_2\| = \sqrt{\frac{1}{4}+\frac{1}{4}+0} = \frac{1}{\sqrt{2}}$

$\vec{v}_3 = \vec{u}_3 - \text{proj}_{W_2}\vec{u}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|}\vec{v}_1 - \dfrac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|}\vec{v}_2$

$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0,0,2)$

$\|\vec{v}_3\| = 2$

$\vec{q}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\vec{q}_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\vec{q}_3 = (0,0,1)$

$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} \langle \vec{q}_1, \vec{u}_1\rangle & \langle \vec{q}_1, \vec{u}_2\rangle & \langle \vec{q}_1, \vec{u}_3\rangle \\ 0 & \langle \vec{q}_2, \vec{u}_2\rangle & \langle \vec{q}_2, \vec{u}_3\rangle \\ 0 & 0 & \langle \vec{q}_3, \vec{u}_3\rangle \end{bmatrix}$

$R = \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 2 \end{bmatrix}$

$QR = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ✓

3 and *Bonus*)
An **inner product** on a real vector space *V* is a function that associates a real number <**u**, **v**> with each pair of vectors **u** and **v** in *V* in such a way that the following axioms are satisfied for all vectors **u**, **v**, and **w** in *V* and all scalars *k*. (+½ pt. each - use *formulas*, *not* the axiom names.)

(1) Symmetry

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

(2) Additivity

$\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(3) Homogeneity

$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

(4) Positivity

$\langle \vec{u}, \vec{u} \rangle \geq 0$ <u>and</u>

$\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$