

# Orthogonal Diagonalization Example — Page 1 of 3

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad \text{First we need to find the eigenvalues and eigenvectors.}$$

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 3 & 2 & -4 \\ 2 & \lambda - 6 & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = (\lambda - 3) \left( \overset{\lambda^2 - 9\lambda + 18 - 4}{(\lambda - 6)(\lambda - 3) - 4} \right) \\ &\quad - 2(2(\lambda - 3) - 8) \\ &\quad - 4(-4 + 4(\lambda - 6)) \\ &= \lambda^3 - 9\lambda^2 + 18\lambda - 4\lambda - 3\lambda^2 + 27\lambda - 54 + 12 \\ &\quad - 4\lambda + 12 + 16 + 16 - 16\lambda + 96 \\ &= \lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda + 2)(\lambda - 7)^2 \end{aligned}$$

Note: You may use technology to help with the algebra for longer polynomials. Be sure you know how to set it all up.

Characteristic Equation:  $(\lambda + 2)(\lambda - 7)^2 = 0$

$\Rightarrow \lambda = -2$  (with algebraic multiplicity 1)

$\lambda = 7$  (with algebraic multiplicity 2)

$\lambda = -2$   
 $(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[ \begin{array}{ccc|c} -5 & 2 & -4 & 0 \\ 2 & -8 & -2 & 0 \\ -4 & -2 & -5 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{matrix} x_1 = -t \\ x_2 = -\frac{1}{2}t \\ x_3 = t \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t$

$\lambda = 7$   
 $(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[ \begin{array}{ccc|c} 4 & 2 & -4 & 0 \\ 2 & 1 & -2 & 0 \\ -4 & -2 & 4 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{matrix} x_1 = t - \frac{1}{2}s \\ x_2 = s \\ x_3 = t \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s$

Eigenspace Basis for  $\lambda = -2$ :  $\left\{ \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$   $\leftarrow$  Geometric Multiplicity 1

Eigenspace Basis for  $\lambda = 7$ :  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$   $\leftarrow$  Geometric Multiplicity 2

$A$  is diagonalizable since the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.

(continued)

Want  $P$  such that  $P^{-1}AP = D$  where  $D$  is diagonal.

$P = [\vec{p}_1 | \vec{p}_2 | \vec{p}_3]$  where the  $\vec{p}_i$  are the eigenvectors of  $A$ .

$$P = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Note: Position of  $\lambda$ s correspond to positions of eigenvectors.

Check:  $P^{-1}AP = \frac{1}{9} \begin{bmatrix} -4 & -2 & 4 \\ 4 & 2 & 5 \\ -2 & 8 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D$

$P^{-1} \rightarrow$

Note: Eigenvectors may be scalar multiples of eigenvectors.

$$\vec{x} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t \rightarrow \text{let } t = -2 \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}, \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s \rightarrow \text{let } t=1, s=2 \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Then  $P = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$  and  $P^{-1}AP = \frac{1}{9} \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 5 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D$

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \text{ is } \underline{\text{orthogonally diagonalizable}} \text{ since } A \text{ is } \underline{\text{symmetric}}.$$

For orthogonal diagonalization,  $P$  must be an orthogonal matrix.

$P^{-1} = P^T \Rightarrow P^T A P = D$  The columns of  $P$  must form an orthonormal set. We need the Gram-Schmidt process.

$A$  is symmetric  $\Rightarrow$  eigenvectors from different eigenspaces are orthogonal.

$$\lambda = -2: \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\} \uparrow \vec{u}_1 \quad \lambda = 7: \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\} \uparrow \vec{u}_2, \uparrow \vec{u}_3$$

$$\vec{u}_1 \cdot \vec{u}_2 = 2 + 0 + -2 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = -2 + 2 + 0 = 0$$

(continued)

Since different eigenspace vectors are already orthogonal, we do not need the Gram-Schmidt process on them all together. We only need to apply it to each separate set.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \underline{\underline{\vec{p}_1}} = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \quad \|\vec{v}_1\| = \sqrt{4+1+4} = 3$$

← Gram-Schmidt on one vector just normalizes it.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\} \Rightarrow \vec{v}_2 = \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \|\vec{v}_2\| = \sqrt{1+0+1} = \sqrt{2}$$

Only applying Gram-Schmidt to two vectors.

$$\vec{v}_3 = \vec{u}_3 - \text{proj}_{\vec{v}_2} \vec{u}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{-1}{(\sqrt{2})^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + \frac{1}{2} \\ 2 + 0 \\ 0 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix} = \vec{v}_3$$

$$\underline{\underline{\vec{p}_2}} = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \underline{\underline{\vec{p}_3}} = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix} -1/3\sqrt{2} \\ 4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix} \quad \|\vec{v}_3\| = \sqrt{\frac{1}{4} + 4 + \frac{1}{4}} = \frac{3}{2}\sqrt{2}$$

$$P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ 1/3 & 0 & 4/3\sqrt{2} \\ -2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$   
 $\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3$

$$P^{-1} = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/3\sqrt{2} & 4/3\sqrt{2} & 1/3\sqrt{2} \end{bmatrix} = P^T$$

$$\text{And } P^T A P = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D$$

All symmetric matrices are orthogonally diagonalizable. However, even if the entries of the matrix are "nice" that does not mean the arithmetic will work out nicely, especially through the Gram-Schmidt process. I found this matrix in an example online and thought that it worked out reasonably well so I used it to develop my example here.

[media.ed.science.psu.edu/sites/media/ed/files/video/5.3v6example3.mp4](http://media.ed.science.psu.edu/sites/media/ed/files/video/5.3v6example3.mp4)