Last Example in Notes: Representations of Functions as Power Series

Ex: $f(x) = \dfrac{x^2}{(1-2x)^2}$ Note: $f(x) = x^2\left(\dfrac{1}{(1-2x)^2}\right)$

Plan: ① $f \to x^2 g$ ← Let's deal with g, then put x^2 back in the end.

②
$$\begin{tikzcd} g \arrow[r, "\mathrm{I}"] & \boxed{\frac{a}{1-r}} \arrow[d] \\ \Sigma \arrow[u, Rightarrow] & \Sigma \arrow[l, "D"'] \end{tikzcd}$$

③ $g \xrightarrow{\text{mult. by } x^2} f$

① $f(x) = x^2\left(\dfrac{1}{(1-2x)^2}\right)$

$\uparrow g(x) = \dfrac{1}{(1-2x)^2}$

$\displaystyle\int g(x)\,dx = \frac{1/2}{1-2x} + C$ How? See *

$a=\frac{1}{2}$, $\underline{\underline{r=2x}}$ $\displaystyle = \left(\sum_{n=0}^{\infty} \frac{1}{2}(2x)^n\right) + C$

$\displaystyle = \left(\sum_{n=0}^{\infty} 2^{n-1}x^n\right) + C$

To get $g(x)$,

$\displaystyle g(x) = \frac{d}{dx}\int g(x)\,dx = \frac{d}{dx}\left[\left(\sum_{n=0}^{\infty} 2^{n-1}x^n\right) + C\right]$

$\displaystyle = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$ ↳ derivative of C is 0

So $\displaystyle g(x) = \sum_{n=1}^{\infty} n 2^{n-1}x^{n-1}$

For f, $\displaystyle f(x) = x^2 g(x) = x^2\sum_{n=1}^{\infty} n 2^{n-1}x^{n-1}$

$\displaystyle = \boxed{\sum_{n=1}^{\infty} n 2^{n-1}x^{n+1} \quad \text{for } R = 1/2}$

Radius of Convergence?

Use r: $r = 2x$

$|r| = |2x| < 1 \Rightarrow -1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

Don't check endpoints if only asking for Radius of Convergence.

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* $\displaystyle\int \frac{1}{(1-2x)^2}\,dx$ $u = 1-2x$, $du = -2\,dx$

$\displaystyle = -\frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}(-u^{-1}) + C = \frac{1}{2}\cdot\frac{1}{u} + C$

$\displaystyle = \frac{1}{2}\cdot\frac{1}{(1-2x)} + C = \frac{1/2}{1-2x} + C$

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