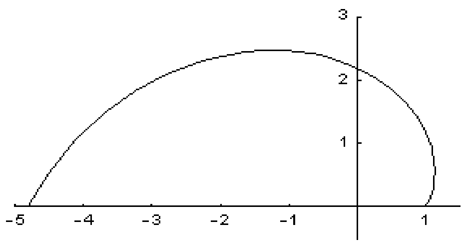
Arc Length in Polar Form

Rectangular: $s=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ use $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$.
Treat θ like t, the parameter. Find $dx/d\theta$ and $dy/d\theta$ then substitute & simplify.

If f is a function with a continuous derivative on $\alpha \le \theta \le \beta$, then the length of the graph of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$ is

$$s=\int_\alpha^\beta \sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example: Find the length of the graph of $r=e^{\theta/2}$ over $0\le\theta\le\pi$.



$r=e^{\theta/2} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}e^{\theta/2}$

$$s=\int_0^\pi \sqrt{(e^{\theta/2})^2+\left(\tfrac{1}{2}e^{\theta/2}\right)^2}\,d\theta$$

$$=\int_0^\pi \sqrt{e^\theta+\tfrac{1}{4}e^\theta}\,d\theta$$

$$=\frac{\sqrt5}{2}\int_0^\pi e^{\theta/2}\,d\theta$$

$$=\sqrt5\, e^{\theta/2}\Big|_0^\pi$$

$$=\underline{\sqrt5\left(e^{\pi/2}-1\right)}$$

Note: Area of Surface of Revolution can also be written in polar form.

$S=2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta$ if rotated about polar axis.

$S=2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta$ if rotated about line $\theta=\frac{\pi}{2}$.